RESEARCHES IN THE LUNAR THEORY.

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Chapter II.

(Continued from p. 147.)

The method of employing numerical values, from the outset, in the equations of condition, determining the a_i, is far less laborious than the literal development of these coefficients in powers of a parameter. For comparison with the results just given, we add the calculation of the coefficients by this method. The following table gives the numerical values of the symbols $[j, i]$, $[j]$ and (j), but the division by the quantity $2(4j^2 - 1) - 4m + m^2$ has been omitted; it is easier to perform this once for all at the end of the series of operations, than to divide each coefficient separately. Hence it must be understood that all the numbers in each department of the table are to be divided by the divisor which stands at the head of it.

Coefficients for a_1 and a_{-1}.

Divisor $= 5.68314\ 08148\ 64695$.

$[1] =$	$0.00861\ 47842\ 96261$	$[-1] = -$	$0.01178\ 75756\ 56865$
$(1) = -$	$0.00623\ 66553\ 18347$	$(-1) = -$	$0.04941\ 95042\ 02516$
$[1, -2] =$	$13.30665\ 60411$	$[-1, -3] = -$	$66.98979\ 68560$
$[1, -1] =$	$6\ 32993\ 22853$	$[-1, -2] = -$	$28.01307\ 31002$
$[1, 2] = -$	$10.71949\ 01593$	$[-1, 1] = -$	$10.96365\ 06556$
$[1, 3] = -$	$15.10904\ 80332$	$[-1, 2] = -$	$38.57409\ 27816$

Coefficients for a_2 and a_{-2}.

Divisor $= 29.68314\ 08148\ 64695$.

$2[2] =$	$0\ 00205\ 43632\ 76229$	$2[-2] = -$	$0.01834\ 79966\ 76898$
$2(2) = -$	$0.02909\ 07097\ 39048$	$2(-2) = -$	$0.07227\ 35586\ 23216$
$[2, -2] =$	$14.97672\ 37558$	$[-2, -4] = -$	$108.69586\ 45706$
$[2, -1] =$	$9.32666\ 40103$	$[-2, -3] = -$	$63.00980\ 48251$
$[2, 1] = -$	$13.00326\ 82750\ 49$	$[-2, -1] = -$	$8.67987\ 25398$

[2, 3] = —	50.03961 76194	[—2, 1] = —	3 64352 31954
[2, 4] = —	74.07269 86888	[—2, 2] = —	19.61044 21261

Coefficients for a_3 and a_{-3}.

Divisor = 69.68314 08149.

[3] = —	0.00113 35729 26473	[—3] = —	0.00793 43596
(3) = —	0.01768 33677	(—3) = —	0.03207 76506 69434
[3, —1] =	12.99224 12519	[—3, —4] = —	114.67538 20668
[3, 1] = —	18.10997 74284	[—3, —2] = —	35.57316 33864
[3, 2] = —	41.33769 10334	[—3, —1] = —	12.34544 97815
[3, 4] = —	103.14632 67728	[—3, 1] =	1.46318 59580

Coefficients for a_4 and a_{-4}.

Divisor = 125.68314 08.

2 [4] = —	0.00428 9733	2 [—4] = —	0.01449 0913
2 (4) = —	0.03864 29156	2 (—4) = —	0.06023 435
[4, —1] =	16.82502 987	[—4, —5] = —	182.50817 069
[4, 1] = —	22.66333 2	[—4, —3] = —	79 01980 9
[4, 2] = —	51.16496 6	[—4, —2] = —	42 51817 5
[4, 3] = —	85.50490 2	[—4, —1] = —	16 17823 8
[4, 5] = —	171.69968 135	[—4, 1] =	6.01654 053

Coefficients for a_5 and a_{-5}.

Divisor = 197.68314.

[5] = —	0.00272 9536	(5) = —	0.02896 299
[5, 1] = —	26.99534 4	[—5, —4] = —	138.68780 0
[5, 2] = —	60.26133 2	[—5, —3] = —	89.42181 0
[5, 3] = —	99.79795 8	[—5, —2] = —	49.88518 4
[5, 4] = —	145.60523 2	[—5, —1] = —	20.07791 2

Coefficients for a_6 and a_{-6}.

Divisor = 285.68314.

2 [6] = —	0.00622 021	2 (—6) = —	0.05640 548
[6, 1] = —	31.21669	[—6, —5] = —	214.46646
[6, 2] = —	68.99224	[—6, —4] = —	152.69091
[6, 3] = —	113.32666	[—6, —3] = —	100.35648

$[6, 4] = -164.21995$ $\qquad$ $[-6, -2] = -\ 57.46319$

$[6, 5] = -221.67212$ $\qquad$ $[-6, -1] = -\ 24.01103$.

These numbers are arranged for carrying the precision to quantities of the 13th order inclusive, and to 15 places of decimals. The quantities $[j, i]$ can be tested by differences, if 0 and the divisor with the negative sign are inserted in the proper places in the series of numbers; for it is evident that the second differences should be constant.

The final results are given below, where, in order that the degree of convergence of this process may be appreciated, we have given the value arising from the first approximation, and then, separately, the corrections arising severally from the second and third approximations. It must be borne in mind that each of these terms is the numerical value, not of an infinite series, but of a rational function of m, and, consequently admits of being computed exact to the last decimal place employed, and, in fact, is here so computed. Hence any error there may be in these values of the a_i arises only from the neglect of the terms of the following approximations, which, in half the number of cases, are of the 14th order, and, in the other half, of the 16th order. It is safe to affirm that these cannot, in any case, exceed two units in the 15th decimal.

	a_1.	a_{-1}.
1st apx, term of 2d order,	$+0.00151\ 58491\ 71593$	$-0.00869\ 58084\ 99634$
2d " " 6th "	$-0.00000\ 01416\ 98831$	$+0.00000\ 00615\ 51932$
3d " " 10th "	$+0.00000\ 00000\ 06801$	$-0.00000\ 00000\ 13838$

$$\frac{a_1}{a_0} = +0.00151\ 57074\ 79563, \quad \frac{a_{-1}}{a_0} = -0.00869\ 57469\ 61540,$$

	a_2.	a_{-2}.
1st apx., term of 4th order,	$+0.00000\ 58793\ 35016$	$+0.00000\ 01636\ 69405$
2d " " 8th "	$-0.00000\ 00006\ 78490$	$+0.00000\ 00001\ 21088$
3d " " 12th "	$+0.00000\ 00000\ 00052$	$-0.00000\ 00000\ 00007$

$$\frac{a_2}{a_0} = +0.00000\ 58786\ 56578, \quad \frac{a_{-2}}{a_0} = +0.00000\ 01637\ 90486,$$

	a_3.	a_{-3}.
1st apx., term of 6th order,	$+0.00000\ 00300\ 35759$	$+0.00000\ 00024\ 60338$
2d " " 10th "	$-0.00000\ 00000\ 04128$	$+0.00000\ 00000\ 00055$

$$\frac{a_3}{a_0} = +0.00000\ 00300\ 31632, \quad \frac{a_{-3}}{a_0} = +0.00000\ 00024\ 60393,$$

	a_4.	a_{-4}.
1st apx., term of 8th order,	$+0.00000\ 00001\ 75296$	$+0.00000\ 00000\ 12284$
2d " " 12th "	$-0.00000\ 00000\ 00028$	$0\ 00000\ 00000\ 00000$

$$\frac{a_4}{a_0} = +0.00000\ 00001\ 75268,\quad \frac{a_{-4}}{a_0} = +0.00000\ 00000\ 12284,$$

Of the 10th order, $\frac{a_5}{a_0} = +0.00000\ 00000\ 01107,\quad \frac{a_{-5}}{a_0} = +0.00000\ 00000\ 00064,$

Of the 12th order, $\frac{a_6}{a_0} = +0.00000\ 00000\ 00007,\quad \frac{a_{-6}}{a_0} = +0.00000\ 00000\ 00000.$

These give the following numerical expression for the coordinates,

$$\begin{aligned} r \cos v = a_0 [1 & - 0.00718\ 00394\ 81977 \ \cos 2\tau \\ & + 0.00000\ 60424\ 47064 \ \cos 4\tau \\ & + 0.00000\ 00324\ 92024 \ \cos 6\tau \\ & + 0.00000\ 00001\ 87552 \ \cos 8\tau \\ & + 0.00000\ 00000\ 01171 \ \cos 10\tau \\ & + 0.00000\ 00000\ 00008 \ \cos 12\tau], \end{aligned}$$

$$\begin{aligned} r \sin v = a_0 [& \quad 0.01021\ 14544\ 41102 \ \sin 2\tau \\ & + 0\ 00000\ 57148\ 66093 \ \sin 4\tau \\ & + 0.00000\ 00275\ 71239 \ \sin 6\tau \\ & + 0.00000\ 00001\ 62985 \ \sin 8\tau \\ & + 0.00000\ 00000\ 01042 \ \sin 10\tau \\ & + 0.00000\ 00000\ 00007 \ \sin 12\tau]. \end{aligned}$$

On comparison of these values with those obtained from the series in $\mathfrak{m}$, the differences are found to be only some units in the 11th decimal.

The coefficients tend to diminish with some regularity as we advance towards higher orders. This is shown by the following scheme of the logarithms and their differences:

	Δ	Δ^2		Δ	Δ^2
n 97.8561			98.0091		
				-3.2521	
94.7812			94.7570		$+9356$
	-2.2694			2.3165	
92.5118		$+307$	92.4405		871
	2.2387			2.2294	
90.2731		341	90.2111		363
	2.2046			2.1931	
88.0685		237	88.0180		201
	2.1809			2.1730	
85.8876			85.8450		

For verification the following equations were computed,

$$\Sigma_i . [(2i+1+\mathrm{m})^2 + 2\mathrm{m}^2] \, \mathrm{a}_i . [\Sigma_i . \mathrm{a}_i]^2 = 1.17141\ 84591\ 84518\ \mathrm{a}_0^3 ,$$
$$\Sigma_i . (-1)^i (2i+1)(2i+1+\mathrm{m}) \, \mathrm{a}_i . [\Sigma_i . (-1)^i \mathrm{a}_i]^2 = 1.17141\ 84591\ 84513\ \mathrm{a}_0^2 .$$

The small difference between the numbers is explained by the fact that, in these formulæ, the quantities a_i are, when i is somewhat large, multiplied by large numbers; as, for instance, a_6 by 169. From the average of these two results we get

$$\mathrm{a}_0 = 0.99909\ 31419\ 75298 \left[\frac{\mu}{n^2}\right]^{\frac{1}{3}} .$$

In the investigations of succeeding chapters, the function $\frac{x}{r^3}$ plays an important part. Hence we will here derive its development as a periodic function of τ by the method of special values. By dividing the quadrant, with reference to τ, into 6 equal parts, we obtain the advantage that the sines or cosines of the multiples of 2τ are either rational or involve $\sqrt{3}$. The special values of the coordinates and of $\frac{x}{r^3}$, thence deduced, are

τ.	$\frac{r}{\mathrm{a}_0} \cos v$.	$\frac{r}{\mathrm{a}_0} \sin v$.	$\frac{x}{r^3}$.
0°	0.99282 60356 45842	0.00000 00000 00000	1.19699 57017 23421
15	0.99378 49245 37167	0.00511 07041 52675	1.19348 68051 03032
30	0.99640 69264 50272	0.00884 83280 32746	1.18399 66676 76716
45	0.99999 39577 40480	0.01021 14268 70906	1.17125 64904 33157
60	1.00358 70309 15127	0.00883 84298 76613	1.15876 77987 29687
75	1.00622 11177 22330	0.00510 08054 31947	1.14978 07679 95764
90	1.00718 60496 23406	0.00000 00000 00000	1.14652 34925 50570.

From the numbers of the last column, by the known process, we deduce

$$\begin{aligned} \frac{x}{r^3} = {} & 1.17150\ 80211\ 79225 \\ & + 0.02523\ 36924\ 97860 \cos 2\tau \\ & + 0.00025\ 15533\ 50012 \cos 4\tau \\ & + 0.00000\ 24118\ 79799 \cos 6\tau \\ & + 0.00000\ 00226\ 05851 \cos 8\tau \\ & + 0.00000\ 00002\ 08750 \cos 10\tau \\ & + 0.00000\ 00000\ 01908 \cos 12\tau \\ & + 0.00000\ 00000\ 00017 \cos 14\tau . \end{aligned}$$

The last coefficient has been added from induction, after which it becomes necessary, as is plain, to subtract an equal quantity from the coefficient of $\cos 10\tau$. Writing the logarithms, as in the former case, we have, the last logarithm being supplied from estimation,

	Δ	Δ^2	Δ^3
98.4020			
	$-$2.0014		
96.4006		$-$168	
	2.0182		$+$68
94.3824		100	
	2.0282		36
92.3542		64	
	2.0346		20
90.3196		44	
	2.0390		10
88.2806		34	
	2.0424		
86.2382			

It will be noticed how much slower this series converges than those for the coordinates.

Any information regarding the motion of satellites having long periods of revolution about their primaries will doubtless be welcome, as the series given by previous investigators are inadequate for showing anything in this direction. Hence this chapter will be terminated by a table of the more salient properties of the class of satellites having the radius vector at a minimum in syzygies and at a maximum in quadratures. For this end I have selected, besides the earth's moon, taken for the sake of comparison, the moons of 10, 9, 8, . . . , 3 lunations in the periods of their primaries, and also what may be called the moon of maximum lunation, as, of the class of satellites under discussion, exhibiting the complete round of phases, it has the longest lunation.

In order that the table may be readily applicable to satellites accompanying any planet, the canonical linear and temporal units, that is those for which μ and n' are both unity, will be used.

From the foregoing methods we obtain :

$$\text{For } \mathrm{m} = \frac{1}{10};$$

$$r \cos v = \mathrm{a}\,[1 - 0.011230 \cos 2\tau + 0.000015 \cos 4\tau], \qquad r \sin v = \mathrm{a}\,[\;0.016102 \sin 2\tau + 0.000014 \sin 4\tau],$$

$$\log \mathrm{a} = 9.3051648.$$

For $\mathrm{m} = \frac{1}{9}$;

$$r \cos v = \mathrm{a}\,[1 - 0.014044 \cos 2\tau + 0.0000247 \cos 4\tau]\,, \qquad r \sin v = \mathrm{a}\,[\;0.020232 \sin 2\tau + 0.0000230 \sin 4\tau]\,,$$

$$\log \mathrm{a} = 9.3326467\,.$$

For $\mathrm{m} = \frac{1}{8}$;

$$r \cos v = \mathrm{a}\,[1 - 0.018061 \cos 2\tau + 0.0000421 \cos 4\tau + 0.00000057 \cos 6\tau]\,, \qquad r \sin v = \mathrm{a}\,[\;0.026172 \sin 2\tau + 0.0000388 \sin 4\tau + 0.00000048 \sin 6\tau]\,,$$

$$\log \mathrm{a} = 9.3630019\,.$$

For $\mathrm{m} = \frac{1}{7}$;

$$r \cos v = \mathrm{a}\,[1 - 0.02407886 \cos 2\tau + 0.00007760 \cos 4\tau + 0.00000141 \cos 6\tau + 0.000000025 \cos 8\tau]\,, \qquad r \sin v = \mathrm{a}\,[\;0.03516059 \sin 2\tau + 0.00007063 \sin 4\tau + 0.00000118 \sin 6\tau + 0.000000022 \sin 8\tau]\,,$$

$$\log \mathrm{a} = 9.3969048\,.$$

For $\mathrm{m} = \frac{1}{6}$;

$$r \cos v = \mathrm{a}\,[1 - 0.03368245 \cos 2\tau + 0.00015943 \cos 4\tau + 0.000004077 \cos 6\tau + 0.000000097 \cos 8\tau]\,, \qquad r \sin v = \mathrm{a}\,[\;0.04968194 \sin 2\tau + 0.00014312 \sin 4\tau + 0.000003393 \sin 6\tau + 0.000000084 \sin 8\tau]\,,$$

$$\log \mathrm{a} = 9.4352928\,.$$

For $\mathrm{m} = \frac{1}{5}$;

$$r \cos v = \mathrm{a}\,[1 - 0.05038803 \cos 2\tau + 0.00038127 \cos 4\tau + 0.000014686 \cos 6\tau + 0.000000505 \cos 8\tau]\,, \qquad r \sin v = \mathrm{a}\,[\;0.07536021 \sin 2\tau + 0.00033582 \sin 4\tau + 0.000012168 \sin 6\tau + 0.000000438 \sin 8\tau]\,,$$

$$\log \mathrm{a} = 9.4795445\,.$$

For $\mathrm{m} = \frac{1}{4}$;

$$r \cos v = \mathrm{a}\,[1 - 0.08331972 \cos 2\tau + 0.00114564 \cos 4\tau + 0.00007409 \cos 6\tau + 0.00000404 \cos 8\tau],$$

$$r \sin v = \mathrm{a}\,[\,0.12709553 \sin 2\tau + 0.00098090 \sin 4\tau + 0.00006099 \sin 6\tau + 0.00000342 \sin 8\tau].$$

$$\log \mathrm{a} = 9.5318013.$$

For $\mathrm{m} = \frac{1}{3}$;

$$r \cos v = \mathrm{a}\,[1 - 0.1622330 \cos 2\tau + 0.0048920 \cos 4\tau + 0.00059858 \cos 6\tau + 0.000081198 \cos 8\tau + 0.000011873 \cos 10\tau + 0.000001849 \cos 12\tau],$$

$$r \sin v = \mathrm{a}\,[\,0.2542740 \sin 2\tau + 0.0039840 \sin 4\tau + 0.00049306 \sin 6\tau + 0.000070196 \sin 8\tau + 0.000010611 \sin 10\tau + 0.0000016902 \sin 12\tau],$$

$$\log \mathrm{a} = 9.5955815.$$

For moons of much longer lunations the methods hitherto used are not practicable, and, in consequence, we resort to mechanical quadratures. Here we shall have two cases. The satellite may be started at right angles to and from a point on the line of syzygies, and the motion traced across the first quadrant; or it may be started at right angles to and from a point on the line of quadratures, and the motion traced across the second quadrant; the prime object being to discover what value of the initial velocity will make the satellite intersect perpendicularly the axis at the farther side of the quadrant.

The differential equations

$$\frac{d^2x}{dt^2} - 2\frac{dy}{dt} + \left[\frac{1}{r^3} - 3\right] x = 0,$$

$$\frac{d^2y}{dt^2} + 2\frac{dx}{dt} + \frac{y}{r^3} = 0,$$

give, as expressions of the values of the coordinates, in the first case,

$$x = x_0 + 2\int_0^t y dt - \int_0^t\!\int_0^t \left[\frac{1}{r^3} - 3\right] x dt^2,$$

$$y = 2\int_0^t (x_0 - x)\, dt - \int_0^t\!\int_0^t \frac{y}{r^2}\, dt^2,$$

and, in the second case,

$$x = -2\int_0^t (y_0 - y)\,dt - \int_0^t\!\int_0^t \left[\frac{1}{r^3} - 3\right] x\,dt^2,$$

$$y = y_0 - 2\int_0^t x\,dt - \int_0^t\!\int_0^t \frac{y}{r^3}\,dt^2.$$

Here the subscript ($_0$) denotes values which belong to the beginning of motion, and ($_1$) will hereafter be used to denote those which belong to the end.

Let v be the velocity, and σ the angle, the direction of motion, relative to the rotating axes, makes with the moving line of syzygies. In the first case then $\sigma = 90°$, and we wish to ascertain what value of v_0 will make $\sigma_1 = 180°$. Generally, for small values of v_0, σ_1 will come out but little less than $270°$; but, as v_0 augments, σ_1 will be found to diminish, and, if x_0 does not exceed a certain limit, a value of v_0 can be found which will make $\sigma_1 = 180°$. In the second case, in like manner, we seek what value of v_0 will make $\sigma_1 = 270°$.

Mechanical quadratures performed with axes of coordinates having no rotation possess some advantages, as, in this case, the velocities are not present in the expressions of the second differentials of the coordinates.

Let X and Y denote the coordinates of the moon in this system, and λ its longitude measured from the line of the last syzygy, from which t is also counted. Then the potential function is

$$\Omega = \frac{1}{r} - \frac{1}{2}r^2 + \frac{3}{2}(X\cos t + Y\sin t)^2.$$

And

$$\frac{d^2X}{dt^2} = \frac{d\Omega}{dX} = -\left[\frac{1}{r^3} + 1\right] X + 3r\cos(\lambda - t)\cos t,$$

$$\frac{d^2Y}{dt^2} = \frac{d\Omega}{dY} = -\left[\frac{1}{r^3} + 1\right] Y + 3r\cos(\lambda - t)\sin t.$$

Therefore, if we compute p and θ from

$$p\cos\theta = -\left[\frac{1}{r^2} - 2r\right]\cos(\lambda - t),$$

$$p\sin\theta = -\left[\frac{1}{r^2} + r\right]\sin(\lambda - t),$$

we shall have

$$\frac{d^2X}{dt^2} = p\cos(\theta + t),$$

$$\frac{d^2Y}{dt^2} = p\sin(\theta + t).$$

The needed values of v and σ can be derived from the equations

$$v \cos(\sigma + t) = \frac{dX}{dt} + Y,$$

$$v \sin(\sigma + t) = \frac{dY}{dt} - X.$$

The developments of the coordinates in ascending powers of t, t being counted from any desired epoch, can often be employed with advantage. Differentiating the differential equations n times we have

$$\frac{d^{n+2}x}{dt^{n+2}} = 2\frac{d^{n+1}y}{dt^{n+1}} + 3\frac{d^n x}{dt^n} - \frac{d^n}{dt^n}(r^{-3}x),$$

$$\frac{d^{n+2}y}{dt^{n+2}} = -2\frac{d^{n+1}x}{dt^{n+1}} - \frac{d^n}{dt^n}(r^{-3}y).$$

Also

$$\frac{d^n}{dt^n}(r^{-3}x) = r^{-3}\frac{d^n x}{dt^n} + n\frac{d(r^{-3})}{dt}\frac{d^{n-1}x}{dt^{n-1}} + \frac{n(n-1)}{1.2}\frac{d^2(r^{-3})}{dt^2}\frac{d^{n-2}x}{dt^{n-2}} + \ldots,$$

with a similar formula for the differential coefficients of $r^{-3}y$.

The differential coefficients of r^{-3}, as far as the 4th, are

$$\frac{d(r^{-3})}{dt} = -3r^{-5}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right),$$

$$\frac{d^2(r^{-3})}{dt^2} = -3r^{-5}\left(x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right) + 15r^{-7}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)^2,$$

$$\begin{aligned}\frac{d^3(r^{-3})}{dt^3} = &-3r^{-5}\left(x\frac{d^3x}{dt^3} + y\frac{d^3y}{dt^3} + 3\frac{dx}{dt}\frac{d^2x}{dt^2} + 3\frac{dy}{dt}\frac{d^2y}{dt^2}\right)\\ &+ 30r^{-7}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)\left(x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right)\\ &- 105r^{-9}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)^3,\end{aligned}$$

$$\begin{aligned}\frac{d^4(r^{-3})}{dt^4} = &-3r^{-5}\left[x\frac{d^4x}{dt^4} + y\frac{d^4y}{dt^4} + 4\frac{dx}{dt}\frac{d^3x}{dt^3} + 4\frac{dy}{dt}\frac{d^3y}{dt^3} + 3\left(\frac{d^2x}{dt^2}\right)^2 + 3\left(\frac{d^2y}{dt^2}\right)^2\right]\\ &+ 45r^{-7}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)\left(x\frac{d^3x}{dt^3} + y\frac{d^3y}{dt^3} + 3\frac{dx}{dt}\frac{d^2x}{dt^2} + 3\frac{dy}{dt}\frac{d^2y}{dt^2}\right)\\ &+ 30r^{-7}\left(x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right)^2\\ &- 525r^{-9}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)^2\left(x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right)\\ &+ 945r^{-11}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)^4.\end{aligned}$$

By means of these formulæ x and y can be expanded in series of ascending powers of t, as far as the term involving t^6, provided we know the values of x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ corresponding to $t=0$. Taking t sufficiently small to make the terms, involving higher powers of t than the sixth, insignificant, as, for instance, $t = 0.05$ or $t = 0.1$, we can ascertain the values of x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at the end of this time. With these values we can again construct new series for x and y in powers of t, in which the latter variable is counted from the end of the previous time. By repetitions of this process the integration can be carried as far as desired. Jacobi's integral, which has not been put to use in the preceding formulæ, can be employed as a check.

In case the body starts from, and at right angles to, either axis, the coefficients of every other power of t in the series for the coordinates vanish.

Thus when the axis in question is that of x, the series for the coordinates have the forms

$$x = x_0 + A_2t^2 + A_4t^4 + A_6t^6 + A_8t^8 + \ldots,$$
$$y = v_0t + A_3t^3 + A_5t^5 + A_7t^7 + A_9t^9 + \ldots.$$

By substitution of these values in the differential equations and the equating of each resulting coefficient to zero we arrive at the following equations;

$$1.2\,A_2 = 2v_0 + 3x_0 - x_0^{-2},$$

$$2.3\,A_3 = -4A_2 - x_0^{-3}v_0,$$

$$3.4\,A_4 = 6A_3 + 3A_2 + \tfrac{1}{2}x_0^{-4}(3v_0^2 + 4x_0A_2),$$

$$4.5\,A_5 = -8A_4 + \tfrac{3}{2}x_0^{-5}v_0(v_0^2 + 2x_0A_2) - x_0^{-3}A_3,$$

$$\begin{aligned}5.6\,A_6 = {} & 10A_5 + 3A_4 + \tfrac{1}{2}x_0^{-4}(6v_0A_3 + 4x_0A_4 + 3A_2^2) \\ & - \tfrac{3}{8}x_0^{-6}(v_0^2 + 2x_0A_2)(5v_0^2 + 6x_0A_2),\end{aligned}$$

$$\begin{aligned}6.7\,A_7 = {} & -12A_6 + \tfrac{3}{2}x_0^{-5}v_0(2v_0A_3 + 2x_0A_4 + A_2^2) - \tfrac{15}{8}x_0^{-7}v_0(v_0^2 + 2x_0A_2)^2 \\ & + \tfrac{3}{2}x_0^{-5}(v_0^2 + 2x_0A_2)A_3 - x_0^{-3}A_5,\end{aligned}$$

$$\begin{aligned}7.8\,A_8 = {} & 14A_7 + 3A_6 + \tfrac{1}{2}x_0^{-4}(6v_0A_5 + 4x_0A_6 + 6A_2A_4 + A_3^2 \\ & - \tfrac{3}{4}x_0^{-6}(v_0^2 + 2x_0A_2)(10v_0A_3 + 8x_0A_4 + A_2^2) \\ & + \tfrac{5}{16}x_0^{-8}(v_0^2 + 2x_0A_2)^2(7v_0^2 + 8x_0A_2) \\ & + \tfrac{3}{2}x_0^{-5}(2v_0A_3 + 2x_0A_4 + A_2^2)A_2,\end{aligned}$$

$$
\begin{aligned}
8.9\,A_9 = &- 16A_8 + \tfrac{3}{2}\, x_0^{-5} v_0 \,(2v_0 A_5 + 2x_0 A_6 + 2A_2 A_4 + A_3^2) \\
&- \tfrac{15}{4}\, x_0^{-7} v_0^2 \,(v_0 + 2x_0 A_2)\,(2v_0 A_3 + 2x_0 A_4 + A_2^2) \\
&+ \tfrac{35}{16}\, x_0^{-9} v_0 \,(v_0^2 + 2x_0 A_2)^3 + \tfrac{3}{2}\, x_0^{-5}\,(2v_0 A_3 + 2x_0 A_4 + A_2^2)\, A_3 \\
&- \tfrac{15}{8}\, x_0^{-7}\,(v_0^2 + 2x_0 A_2)^2 A_3 + \tfrac{3}{2}\, x_0^{-5}\,(v_0^2 + 2x_0 A_2)\, A_5 - x_0^{-3} A_7 \,.
\end{aligned}
$$

By means of these relations each A can be derived from all the A which precede it.

When the axis is that of y, the series have the forms

$$
\begin{aligned}
x &= v_0 t + A_3 t^3 + A_5 t^5 + A_7 t^7 + A_9 t^9 + \ldots\,, \\
y &= y_0 + A_2 t^2 + A_4 t^4 + A_6 t^6 + A_8 t^8 + \ldots\,.
\end{aligned}
$$

And the equations, determining the coefficients A, are

$$
\begin{aligned}
1.2\,A_2 &= - 2v_0 - y_0^{-2}\,, \\
2.3\,A_3 &= 4A_2 + 3v_0 - y_0^{-3} v_0\,, \\
3.4\,A_4 &= - 6A_3 + \tfrac{1}{2}\, y_0^{-4}\,(3v_0^2 + 4y_0 A_2)\,, \\
4.5\,A_5 &= 8A_4 + 3A_3 + \tfrac{3}{2}\, y_0^{-5}\, v_0\,(v_0^2 + 2y_0 A_2) - y_0^{-3} A_3\,.
\end{aligned}
$$

The equations are not written as far as in the former case, as it is evident they may be derived from the preceding group by putting y_0 in the place of x_0, reversing the signs of the first terms, and removing the term $3A_{n-2}$ from the equations, which give the values of the A of even subscripts, into those which give the values of the A of odd subscripts, after having augmented the subscript by unity.

The velocity of the moon of maximum lunation vanishes in quadratures, and when $v_0 = 0$ the preceding series become, putting $y_0^{-3} = \alpha$,

$$
\begin{aligned}
x = y_0 \Big[&- \frac{1}{3}\,\alpha t^3 + \Big(\frac{1}{60}\,\alpha - \frac{1}{60}\,\alpha^2\Big)\, t^5 + \Big(- \frac{1}{2520}\,\alpha + \frac{1}{315}\,\alpha^2 + \frac{1}{280}\,\alpha^3\Big)\, t^7 \\
&+ \Big(\frac{1}{181440}\,\alpha - \frac{1}{12096}\,\alpha^2 + \frac{1}{45360}\,\alpha^3 + \frac{47}{9072}\,\alpha^4\Big)\, t^9 \\
&+ \Big(\frac{1}{19958400}\,\alpha + \frac{317}{9979200}\,\alpha^2 - \frac{13}{120960}\,\alpha^3 - \frac{10403}{4989600}\,\alpha^4 + \frac{947}{237600}\,\alpha^5\Big)\, t^{11} \Big]\,, \\
y = y_0 \Big[&1 - \frac{1}{2}\,\alpha t^2 + \Big(\frac{1}{6}\,\alpha - \frac{1}{12}\,\alpha^2\Big)\, t^4 + \Big(- \frac{1}{180}\,\alpha + \frac{1}{60}\,\alpha^2 - \frac{11}{360}\,\alpha^3\Big)\, t^6 \\
&+ \Big(\frac{1}{10080}\,\alpha - \frac{1}{1008}\,\alpha^2 + \frac{13}{1120}\,\alpha^3 - \frac{73}{5040}\,\alpha^4\Big)\, t^8 \\
&+ \Big(- \frac{1}{907200}\,\alpha + \frac{17}{90720}\,\alpha^2 - \frac{1}{756}\,\alpha^3 + \frac{4603}{453600}\,\alpha^4 - \frac{3}{400}\,\alpha^5\Big)\, t^{10} \Big]\,.
\end{aligned}
$$

These series suffice for computing the values of x and y with the desired exactitude when t is less than 0.3.

This special case of the moon of maximum lunation will now be treated. As there seems to be no ready method of getting even a roughly approximate value of y_0, we are reduced to making a series of guesses. I first took $y_0 = 0.82$; tracing the path to its intersection with the axis of x, σ_1, which ought to be 270°, came out 261° 29′ 47″.9. A second trial was made with $y_0 = 0.7937$; the result was $\sigma_1 = 267° 37' 8''.3$. Again a third trial with $y_0 = 0.7835$ gave $\sigma_1 = 269° 41' 13''.3$. The principal data acquired in the three trials are given in the following lines:

y_0.	T.	x_1.	$\frac{dx_1}{dt}$.	$\frac{dy_1}{dt}$.	σ_1.	Maximum Variation.
0.8200	0.972430	—0.339523	—0.288149	—1.927275	261° 29′ 47″.9	44° 57′ 4″
0.7937	0.908207	—0.290945	—0.089184	—2.144832	267 37 8 .3	46 39 36
0.7835	0.884782	—0.274324	—0.012170	—2.227928	269 41 13 .3	47 17 21 .

T denotes the time employed in crossing the quadrant, and the last column contains the maximum value of the angular deviation of the body from its mean direction as seen from the origin, that is, the direction it would have had, had it moved across the quadrant with a uniform angular velocity about the origin.

A check may be had on the accuracy of the computations by mechanical quadratures. We determine the value of the constant $2C$ which completes Jacobi's integral from the coordinates and velocities, both at the beginning and at the end of the motion, for each of the three trials. The result is

y_0.	First value.	Second value.
0.8200	2.34902	2.43901
0.7937	2.51985	2.51987
0.7835	2.55265	2.55261 .

We can now apply Lagrange's general interpolation formula to these data, and, regarding σ_1 as the independent variable, inquire what are the values which correspond to $\sigma_1 = 270°$. The numbers of the first trial must be multiplied by $+0.014861$; those of the second by -0.210190; those of the third by $+1.195329$, and the sums taken. The results are

y_0.	T.	x_1.	$\frac{dx_1}{dt}$.	$\frac{dy_1}{dt}$.	$2C$.	Maximum Variation.
0.781898	0.881160	0.271798	—0.000083	—2.24093	2.55788	47° 23′ 12″.

That $\frac{dx_1}{dt}$ does not rigorously vanish is due to the employment of only three terms in the interpolation; for the same reason the value of $2C$ does not quite agree with that obtained from the values of x_1 and $\frac{dy_1}{dt}$. To make all these elements accordant we add 0.00009 to the value of $\frac{dy_1}{dt}$.

A table of approximate values of x and y, derived roughly from the data afforded by the process of mechanical quadratures is appended: they will serve for plotting the orbit.

t.	x.	y.	t.	x.	y.	t.	x.	y.
0.00	—.0000	+.7819	0.30	—.0148	+.7080	0.60	—.1177	+.4748
0.02	.0000	.7816	0.32	.0180	.6978	0.62	.1294	.4519
0.04	.0000	.7806	0.34	.0215	.6869	0.64	.1418	.4277
0.06	.0001	.7790	0.36	.0256	.6752	0.66	.1547	.4022
0.08	.0003	.7767	0.38	.0301	.6629	0.68	.1680	.3752
0.10	.0005	.7737	0.40	.0351	.6499	0.70	.1818	.3466
0.12	.0009	.7701	0.42	.0407	.6361	0.72	.1956	.3162
0.14	.0015	.7659	0.44	.0468	.6216	0.74	.2095	.2839
0.16	.0022	.7610	0.46	.0534	.6063	0.76	.2230	.2496
0.18	.0032	.7554	0.48	.0607	.5902	0.78	.2359	.2131
0.20	.0044	.7492	0.50	.0686	.5733	0.80	.2475	.1745
0.22	.0058	.7432	0.52	.0771	.5555	0.82	.2575	.1339
0.24	.0076	.7347	0.54	.0863	.5369	0.84	.2653	.0913
0.26	.0096	.7265	0.56	.0961	.5172	0.86	.2704	.0474
0.28	.0120	.7176	0.58	.1066	.4965	0.88	.2718	.0027

The following is the table of the numerical values of the quantities of principal interest belonging to the moons mentioned at the beginning of this paragraph. In the first line stands the earth's moon, having very approximately $12\frac{59}{160}$ lunations in the period of its primary. In the last line is the moon of maximum lunation. The quantities belonging to the moon of two lunations have been somewhat rudely inferred from the numbers in the adjacent lines.

Number of Lunations in period of Primary. $\frac{1}{m}$.	Radius Vector in Syzygies. r_0.	Radius Vector in Quadratures. r_1.	Ratio. $\frac{r_1}{r_0}$.	Velocity in Syzygies. v_0.	Velocity in Quadratures. v_1.	Ratio. $\frac{v_1}{v_0}$.	$2C$.	Maximum Variation.
$12\frac{59}{160}$	0.17610	0.17864	1.01446	2.22295	2.16484	0.97386	6.50888	0° 35′ 6″
10	0.19965	0.20418	1.02271	2.06163	1.97693	0.95892	5.88686	0 55 21
9	0.21209	0.21813	1.02849	1.98730	1.88501	0.94853	5.61562	1 9 33
8	0.22652	0.23485	1.03678	1.90904	1.78250	0.93372	5.33873	1 29 58
7	0.24342	0.25543	1.04934	1.82721	1.66572	0.91162	5.05535	2 0 53
6	0.26332	0.28167	1.06969	1.74333	1.52851	0.87677	4.76409	2 50 49
5	0.28660	0.31699	1.10605	1.66247	1.35953	0.81777	4.46103	4 18 37
4	0.31232	0.36897	1.18138	1.60111	1.13480	0.70876	4.13277	7 17 0
3	0.33235	0.45973	1.38329	1.62141	0.79387	0.48962	3.72018	14 34 14
2	0.302	0.684	2.26	2.00	0.18	0.09	2.89	37 21
1.78265	0.27180	0.78190	2.87676	2.24102	0.00000	0.00000	2.55788	47 23 12

In regard to this table we may notice the following points. The moon of the last line is the most remarkable: it is, of the class of satellites considered in this chapter, (viz., those which have the radius vector at a minimum in syzygies, and at a maximum in quadratures,) that which, having the longest lunation, is still able to appear at all angles with the sun, and thus undergo all possible phases. Whether this class of satellites is properly to be prolonged beyond this moon, can only be decided by further employment of mechanical quadratures. But it is at least certain that the orbits, if they do exist, do not intersect the line of quadratures, and that the moons describing them would make oscillations to and fro, never departing as much as 90° from the point of conjunction or of opposition.

This moon is also remarkable for becoming stationary with respect to the sun when in quadrature; and its angular motion near this point is so nearly equal to that of the sun that, for about one-third of its lunation, it is within 1° of quadrature. From the data of the table we learn that such a moon, circulating about the earth, would make a lunation in 204.896 days.

We notice that the radius vector in syzygies of this class of satellites arrives at a maximum before we reach the moon of maximum lunation. This

maximum value is very nearly, if not exactly, $\frac{1}{3}$, when measured in terms of our linear unit, and thus is a little less than double the radius vector of the earth's moon. It occurs in the case of the moon which has about 2.8 lunations in the period of its primary.

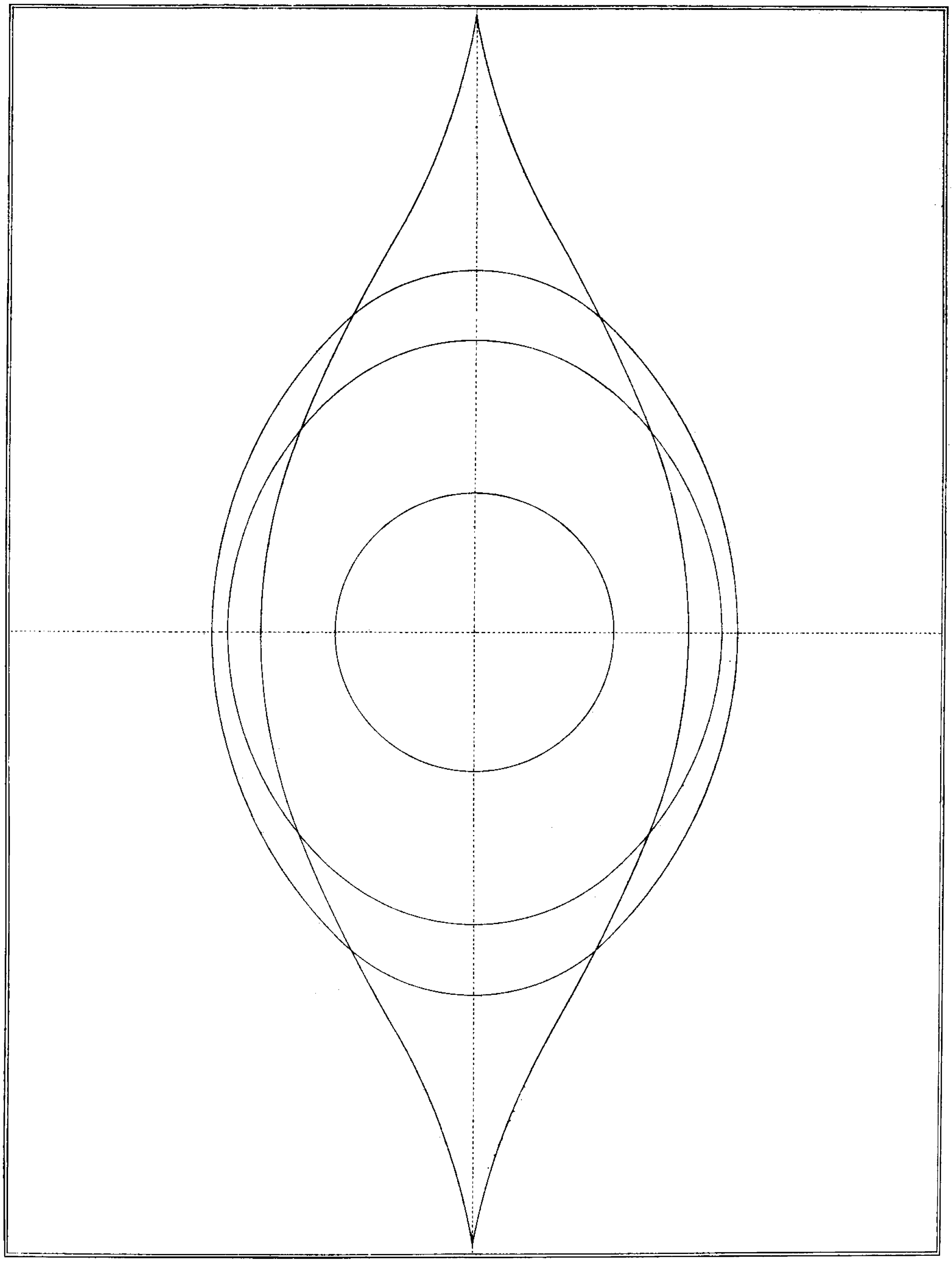
The radius vector in quadratures augments continuously as the length of the lunation increases, as also does the ratio of these radii, until, in the moon of maximum lunation, the radius in quadratures is but little less than three times that in syzygies.

The velocity in syzygies does not continuously diminish, but attains a minimum somewhere about the moon of four lunations, and afterwards augments so that, for the moon of maximum lunation, it does not differ greatly from the velocity of the earth's moon in syzygies. On the other hand the velocity in quadratures constantly diminishes.

The maximum value of the variation augments rapidly with increase in the length of lunation, so that, in the moon of maximum lunation, it exceeds an octant, or is more than 80 times the value which belongs to the earth's moon.

In the adjoining figure are constructed graphically the paths of the earth's moon, of the moons of four and three lunations, and of the moon of maximum lunation. The moons in the first lines of the table have paths which approach the ellipse quite closely, but the paths of the moons of the last lines exhibit considerable deviation from this curve, while the orbit of the moon of maximum lunation has sharp cusps at the points of quadrature.

(To be continued.)



HILL, *Researches in the Lunar Theory.*